ON PLIABILITY OF DEL PEZZO FIBRATIONS AND COX RINGS

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Abstract. We develop some concrete methods to work with low rank Cox rings. As part of this development, we give an algorithm to construct explicitly the coarse moduli space of a toric Deligne-Mumford stack. This can be viewed as the generalization of the notion of well-formedness for weighted projective spaces to homogeneous coordinate ring of toric varieties. Then we apply these methods to study birational transformations of certain fibrations of del Pezzo surfaces over $\mathbb{P}^1$, into other Mori fibre spaces, using Cox rings and variation of geometric invariant theory. We show that the pliability of these Mori fibre spaces is at least three and they are not rational.

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1. Introduction

This article is organised in two parts. In the first part we develop some techniques to work with low rank Cox rings and their blow ups. In particular, we show how to write down the Cox ring of the (weighted) blow up of a weighted projective space and also the (weighted) blow up of the projectivization of a weighted vector bundle over $\mathbb{P}^n$. We also introduce a notion of well-formedness for toric Cox rings, which gives rise to an algorithm of constructing the coarse moduli space of a toric Deligne-Mumford stack.

In the second part, we apply these techniques to study birational geometry of certain families of del Pezzo surfaces, treated as Mori fibre spaces. Mori fibre spaces are outcomes of the minimal model program for varieties with negative Kodaira dimension. Formally, a variety X with at worst $\mathbb{Q}$-factorial terminal singularities is a Mori fibre space if there exists a morphism $\varphi\colon X \to Z$ to a normal variety Z, of strictly smaller dimension than X, such that $-K_X$ is φ-ample and the relative Picard number is equal to one, that is $\rho(X/Z) = 1$. It is crucial in the classification of threefolds to investigate how many different Mori fibre spaces fall in the same birational class, and study their properties. Naturally, this problem is considered up to *square birational* equivalence.

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Definition 1.1. Let $X \to Z$ and $X' \to Z'$ be Mori fibre spaces. A birational map $f\colon X \dashrightarrow X'$ is *square* if it fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

where g is birational and, in addition, the map $f_L\colon X_L \dashrightarrow X'_L$ induced on generic fibres is biregular. In this case we say that X/Z and X'/Z' are *square birational*. We denote this by $X/Z \sim X'/Z'$.

Definition 1.2. The *pliability* of a Mori fibre space $X \to S$ is the set

$$\mathcal{P}(X/S) = \{\text{Mfs } Y \to T \mid X \text{ is birational to } Y\}/\sim$$

We sometimes use the term pliability to mean the cardinality of this set, when it is finite.

For many Mori fibre spaces this number is known to be very small or the variety is known to be rational, so that $\mathcal{P}(X/Z) = \mathcal{P}(\mathbb{P}^n)$ is infinite. For instance smooth fibrations of del Pezzo surfaces over $\mathbb{P}^1$ with degree of the general fibre five or above are known to be rational. On the opposite side general Fano hypersurfaces in a weighted projective space are birationally rigid [10], hence have pliability one. Many other similar results have been obtained for threefold Mori fibre spaces, in both directions. An interesting model with pliability exactly two is constructed as a quartic in $\mathbb{P}^4$ with a cA_2 singular point [9].

In this article we consider a del Pezzo fibration of degree 4 over $\mathbb{P}^1$ and show that its pliability is at least 3. Then using results from [2] we prove it is not rational. We recall that the degree of a del Pezzo surface is the self-intersection number of its canonical class. Birational geometry of del Pezzo fibrations, considered as Mori fibre spaces, plays an important role in the theory of classification of algebraic varieties in dimension 3. While degree 5 or above imply rationality, one expects more rigidity as the degree drops. There are many results of this type on birational rigidity and nonrigidity for degree 1, 2 and 3 fibrations, see for example [1, 3, 5, 16, 20]. The rationality problem in degree 4 is studied in [2, 22]. It is also shown in [2] that these varieties are birational to conic bundles over the base, hence nonrigidity can be deduced from this. We study the birational behaviour of these varieties modulo this manoeuvre.

The construction of the links between various models, through 2-ray games that generate elementary Sarkisov links, are obtained via Cox embeddings. In [4] it was shown how to run type III or IV elementary Sarkisov links on a rank 2 Cox rings. This method has been applied to many classes of Mori fibre spaces with Picard number 2, see [1, 3]. Some of the models we construct are obtained in this fashion and for others we introduce more general methods of working with higher rank Cox rings and also explain what the maps between these Cox rings look like. In other words, these constructions provide explicit Sarkisov links of type I or II in terms of Cox data. The following theorem is the main result of this article concerning birational geometry of threefolds.

Theorem 1.3. *Let $\mathcal{F}$ be the five-fold $\mathcal{F} = \operatorname{Proj}_{\mathbb{P}^1} \mathcal{E}$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(3)$. Let M be the class of the tautological bundle on F and L the class of a fibre (over $\mathbb{P}^1$). Then there are hypersurfaces $Q_1 \in |-3L+2M|$ and $Q_2 \in |-2L+2M|$ in $\mathcal{F}$ such that $X = Q_1 \cap Q_2$ is a complete intersection and*

(1) X is smooth, $\operatorname{Pic}(X) \cong \mathbb{Z}^2$ and $X \to \mathbb{P}^1$ is a Mori fibre space with generic fibre del Pezzo surface of degree 4,

(2) there exist at least two non-trivial elementary Sarkisov links from $X/\mathbb{P}^1$ to other Mori fibre spaces,
(3) X is not rational.

Conditions on the generality of Q_1 and Q_2 are specified in Section 4.

The structure of the article is as follows. In Sections 2 we study well-fomedness of the homogeneous coordinate ring of toric varieties. In Section 3 we show how to work with blow ups of low rank Cox rings. Sections 2 and 3 can be read independently of this article and the results are quite general and do not restrict only to the cases studied in this article. Among applications of these methods are the description of the starting point of type I and II Sarkisov links, that we apply. Section 2 explains how some of the blow up varieties can be modified to simpler ones, isomorphically. This generalizes the notion of well-formedness of weighted projective spaces [18] to that of Cox rings. Equivalently, it is an explicit method to find Cox rings of coarse moduli of toric Deligne-Mumford stacks [14]. In Section 4 we describe explicitly the links between varieties under study. Tools provided in earlier sections will be used frequently in the proofs.

All varieties and stacks in this article are projective. Results in Sections 2 and 3 hold in any field of characteristic zero. In Section 4 varieties are considered over the field of complex numbers.

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2. Well-formedness and stacky models

Weighted projective spaces have been studied extensively, and the well-formedness property, as in [13] and [18] plays an important role in the basic theory.

A weighted projective space, denoted by $\mathbb{P}(a_0, \ldots, a_n)$, for positive integers $a_0, \ldots, a_n$, is defined by the geometric quotient of $\mathbb{C}^{n+1} - \{0\}$ when acted on by $\mathbb{C}^*$ via

$$\lambda.(x_0, \ldots, x_n) \mapsto (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n), \quad \text{for } \lambda \in \mathbb{C}^*.$$

In other words, $\mathbb{P}(a_0, \ldots, a_n) = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$, where $\mathbb{C}[x_0, \ldots, x_n]$ is $\mathbb{Z}$-graded with $\deg(x_i) = a_i$.

The weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is *well formed* if $\gcd(a_0, \ldots, \hat{a_i}, \ldots, a_n) = 1$ for all $0 \le i \le n$. The well-formed model of a given quotient $\mathbb{P}(a_0, \ldots, a_n)$ is obtained in two steps (see [18]):

(1) Remove generic stabilisers: Find $a = \gcd(a_0, \ldots, a_n)$, then replace $\mathbb{P}(a_0, \ldots, a_n)$ by $\mathbb{P}(b_0, \ldots, b_n)$, where $b_i = \frac{a_i}{a}$.
(2) Remove quasi-reflections: For each b_i find $\mathbf{b_i} = \gcd(b_0, \ldots, \hat{b_i}, \ldots, b_n)$ and replace $\mathbb{P}(b_0, \ldots, b_n)$ by
$$\mathbb{P}(\frac{b_0}{\mathbf{b_i}}, \ldots, \frac{b_{i-1}}{\mathbf{b_i}}, b_i, \frac{b_{i+1}}{\mathbf{b_i}}, \ldots, \frac{b_n}{\mathbf{b_i}}).$$

Remark 1. In the setting of [14], the variety obtained at Step 1 above is the toric orbifold associated to the toric Deligne-Mumford stack $\mathbb{P}(a_0, \ldots, a_n)$ and the variety produced at the end (the well-formed model) is the corresponding canonical model, see [14] Example 7.27.

Note that for a given set of positive integers defining the weights, and positive integers α and β the following holds ([18] §5):

$$\mathbb{P}(a_0, \ldots, a_n) \cong \mathbb{P}(\alpha a_0, \ldots, \alpha a_n) \cong \mathbb{P}(a_0, \beta a_1, \ldots, \beta a_n)$$

and this is exactly why one is permitted to do the process above and obtain isomorphic quotients. Our aim in this section is to obtain similar construction for projective toric Deligne-Mumford stacks. These (non-well formed) Cox rings arise naturally in our description of blow ups of Cox rings in the following sections.

Let $\mathfrak{X}$ be a toric stack of dimension d determined by the fan Δ in $N \cong \mathbb{Z}^d$ and denote the set of 1-dimensional cones in Δ by $\Delta(1)$. Assume $M = \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice and $|\Delta(1)| = n$. One has the following exact sequence, see ([15], §3.4).

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^n \xrightarrow{A} \mathrm{Cl}(\mathfrak{X}) \longrightarrow 0 \tag{1}$$

The stack $\mathfrak{X}$ can be realized as the quotient $(\mathbb{C}^n - V(I))/\!/G$, where I, the *irrelevant ideal*, is the ideal of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ obtained from the combinatorial structure of the fan Δ, and $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\mathfrak{X}), \mathbb{C}^*)$. In particular, G is reductive and we have

$$G = (\mathbb{C}^*)^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}, \quad \text{for some } r, k, a_i \in \mathbb{N} \text{ and } r = n - d \quad .$$

Without loss of generality we can assume the torsion part, i.e. $\bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}$, induces no generic stabilisers nor quasi-reflections. Otherwise removing the corresponding part should be easy and produces isomorphic quotients as desired.

The aim is to identify inappropriate components of the non-torsion part that cause such behaviour and *remove* them. Let us assume, for now, for simplicity in writing, that $\mathrm{Cl}(T)$ is torsion free; in other words $k = 0$. Hence $\mathrm{Cl}(\mathfrak{X}) \cong \mathbb{Z}^r$ and it follows that $A \in \mathcal{M}_{r\times n}(\mathbb{Z})$. By applying the functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to the short exact sequence (1) one obtains

$$1 \longrightarrow G \xrightarrow{A^*} (\mathbb{C}^*)^n \longrightarrow \mathbb{T} \longrightarrow 1 \quad , \tag{2}$$

where $\mathbb{T}$ is the torus acting on $\mathfrak{X}$. The action of G on $\mathbb{C}^n$ is the extension of the action on $(\mathbb{C}^*)^n$ above and is identified by the matrix $A = (a_{ij})$ above in the following way

$$(\lambda_1, \ldots, \lambda_r) \cdot x_j \mapsto \prod_{i=1}^{r} \lambda_i^{a_{ij}} x_j \quad .$$

Notation 2.1. We use $\mathfrak{X} = \mathfrak{X}(I, A)$ to denote the quotient of $(\mathbb{C}^n - Z)/\!/G$, where I is the irrelevant ideal, $Z = V(I) \subset \mathbb{C}^n$, $G = (\mathbb{C}^*)^r$ and the action is identified by the matrix $A \in \mathcal{M}_{r\times n}(\mathbb{Z})$ as before.

Definition 2.2. Let $\mathfrak{X} = \mathfrak{X}(I, A)$ be a toric stack. We define the rank of $\mathfrak{X}$ to be $r = \mathrm{rank}\, A$.

Lemma 2.3. *Let $\mathfrak{X} = \mathfrak{X}(I, A)$ and $B = gA$ for some $g \in \mathrm{GL}(r, \mathbb{Q})$ with integer entries and define $\mathfrak{X}'$ to be the toric stack $\mathfrak{X}' = \mathfrak{X}(I, B)$. Then $\mathfrak{X}$ is isomorphic to $\mathfrak{X}'$ as quotient varieties.*

Proof. We give an explicit and set theoretic proof. $\mathfrak{X}$ and T' are defined by

$$\mathfrak{X} = (\mathbb{C}^n - V(I))/G_A \qquad , \qquad \mathfrak{X}' = (\mathbb{C}^n - V(I))/G_B \quad ,$$

where $G_A \cong G_B \cong (\mathbb{C}^*)^r$. If we denote $A = (a_{ij})$ and $B = (b_{ij})$, then for $(\lambda_1, \ldots, \lambda_r) \in G_A$ and $(\gamma_1, \ldots, \gamma_r) \in G_B$, the actions are the following:

$$G_A : \qquad (\lambda_1, \ldots, \lambda_r).(x_1, \ldots, x_n) \mapsto (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n)$$

$$G_B : \qquad (\gamma_1, \ldots, \gamma_r).(x_1, \ldots, x_n) \mapsto (\prod_{i=1}^{r} \gamma_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \gamma_i^{a_{in}} x_n)$$

Let (x) and (y) be two vectors in $\mathbb{C}^n$. Let us denote by (x) $\sim_A$ (y) if (x) and (y) are in the same orbit of the action by G_A, and similarly for (x) $\sim_B$ (y). The aim is to show

$$(\mathrm{x}) \sim_A (\mathrm{y}) \quad \text{if and only if} \quad (\mathrm{x}) \sim_B (\mathrm{y}) \quad .$$

If (x) $\sim_B$ (y), then there exists $(\gamma_1, \ldots, \gamma_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \ldots, y_n) = (\prod_{i=1}^{r} \gamma_i^{b_{i1}} x_1, \ldots, \prod_{i=1}^{r} \gamma_i^{b_{in}} x_n) \quad .$$

To prove (x) $\sim_A$ (y), we must find $(\lambda_1, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$ such that

$$(y_1, \ldots, y_n) = (\prod_{i=1}^{r} \lambda_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^{r} \lambda_i^{a_{in}} x_n) \quad .$$

This follows from $b_{ij} = \sum_k g_{ik} a_{kj}$, if we put $\lambda_i = \gamma_1^{g_{i1}} \ldots \gamma_r^{g_{ir}}$.
Proof for the only if part is very similar and it is done by replacing g by g^{-1}. □

The result of Lemma 2.3 shows that the expression $\mathfrak{X}(I, A)$ is not uniquely determined from A, when considered as varieties, and it varies up to the action of a subset of $\mathrm{GL}(r, \mathbb{Q})$. In fact failure of this set to be a subgroup is the problem of well-formedness. In the rest of this section, we complete our task of finding a well formed model for $\mathfrak{X}(I, A)$. Furthermore, it will be noted that such a model is unique up to $\mathrm{SL}^*(r, \mathbb{Z})$, the group of integer matrices with determinant ± 1.

Definition 2.4. Let $M \in \mathcal{M}_{r\times n}(\mathbb{Z})$ be a rank r matrix ($r < n$). Suppose $m_1, \ldots, m_s$ are all the non-zero $r \times r$ minors of M and let $d_M = \gcd(|m_1|, \ldots, |m_k|)$. The matrix M is called *standard* if $d_M = 1$.

Lemma 2.5. *For any rank r matrix $M \in \mathcal{M}_{r\times n}(\mathbb{Z})$, there exist matrices $g \in \mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r\times r}(\mathbb{Z})$ and $N \in \mathcal{M}_{r\times n}(\mathbb{Z})$ such that $M = gN$ and N is a standard matrix of rank r.*

We try to remove every factor of d_M by multiplying M with a matrix whose inverse is in $\mathrm{GL}(r, \mathbb{Q}) \cap \mathcal{M}_{r\times r}(\mathbb{Z})$.

Proof. If $d_M = 1$, then there is nothing to prove. Assume p is a prime factor of d_M and m is the biggest integer for which $p^m \mid d_M$. If p^k (for some positive k) divides every entry of the first row of M then multiply M with an $r \times r$ diagonal matrix $H = (h_{ij})$ with $h_{ii} = 1$ for $i > 1$ and $h_{11} = \frac{1}{p^k}$. It is obvious that $M^{(1)} = HM \in \mathcal{M}_{r\times n}(\mathbb{Z})$ and $d_M = p^k d_{M^{(1)}}$.
If $k = m$ we have managed to remove p^m as it was promised. Now assume $k < m$ and let $M^{(1)} = (a_{ij})$. There is at least one non-zero entry in the first row of $M^{(1)}$ which is not divisible by p. Without loss of generality we can assume this entry is a_{11}. By assumption, there is another non-zero entry in the first row. Assume, without loss of generality, that a_{21} is non-zero and suppose $\gcd(a_{11}, a_{21}) = a$, then there exist integers b and c such that $ba_{11} + ca_{21} = a$. Let H_1 be the following matrix:

$$\begin{pmatrix} b & c & 0 & \cdots & 0 \\ -\frac{a_{21}}{a} & \frac{a_{11}}{a} & & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The matrix $M^{(2)} = H^1 M^{(1)}$ has the following shape

$$\begin{pmatrix} a & * & \cdots \\ 0 & * & \cdots \\ * & * & \\ \vdots & & \ddots \end{pmatrix} .$$

Obviously $\det(H^1) = 1$ and a is not divisible by p.

By repeating this process for all entries of the first column, except a_{11}, we can replace them by 0. Now let $M_1^{(2)}$ be the $(r-1)\times(n-1)$ sub-matrix of $M^{(2)}$ obtained by removing the first row and column. Obviously $\det(M^{(2)}) = a.\det(M_1^{(2)})$. This forces p^{m-k} to divide $\det(M_1^{(2)})$.

We can repeat the algorithm above and remove all powers of p from the first row of $M_1^{(2)}$. If there is any factor of p left, we apply the process above to the second column of the new matrix to make its entries equal to zero.

By repeating this algorithm we find a matrix M' for which $d_M = p^m \times d_{M'}$. All these can be done again for a prime factor of $d_{M'}$. After finitely many steps we will have a matrix N with $d_N = 1$. □

Remark 2. This process can be thought of as a special version of the Smith normal form without column operations, and with an emphasis on keeping track of the number d_A at each step.

Corollary 2.6. *For any $\mathfrak{X}(I,A)$, there exists a standard matrix B such that $\mathfrak{X}(I,A) \cong \mathfrak{X}(I,B)$.*

Proof. This follows from Lemma 2.3 and Lemma 2.5. □

Proposition 2.7. *Let $A \in \mathcal{M}_{r\times n}$ be a matrix of rank r. The following are equivalent.*

(i) $A\colon \mathbb{Z}^n \to \mathbb{Z}^r$ *is surjective.*
(ii) $\wedge^r A\colon \wedge^r \mathbb{Z}^n \to \wedge^r \mathbb{Z}^r \cong \mathbb{Z}$ *is surjective.*
(iii) A *is standard.*

Proof. This is straightforward linear algebra. □

Definition 2.8. Let A be a standard $r\times n$ matrix with integer entries. Suppose A_k is an $r\times(n-1)$ matrix obtained by removing the k-th column of A. The matrix A is called *well formed* if every A_k $(1 \le k \le n)$ is standard.

Lemma 2.9. *Let $\mathfrak{X}(I,A)$ be a toric stack defined by an irrelevant ideal I and an $r\times n$ matrix $A = (a_{ij})$. Assume $q \neq 1$ is a positive integer such that $q \mid a_{1j}$ for $j > 1$ but $q \nmid a_{11}$. Define the matrix $B = (b_{ij})$ by $b_{i1} = q.a_{i1}$ and $b_{ij} = a_{ij}$ for $j > 1$. Then $\mathfrak{X}(I,A) \cong \mathfrak{X}(I,B)$ as quotient varieties.*

Proof. Pick an ample divisor D such that

$$\mathfrak{X} = \operatorname{Proj} R_D \quad ,$$

where R is the Cox ring of $\mathfrak{X}$ generated by $x_1, \ldots, x_n$ with degrees correspond to C_i, columns of the matrix A, and D has degree $D = \sum \alpha_i C_i$, where α_i are non-negative integers. Note that we associate D with its degree D and use the same notation for both.

The ring R_{qD} consists of invariant sections of multiples of qD, i.e.

$$R_{qD} = \left(\bigoplus_{j=0}^{\infty} H^0(\mathbb{C}^d, \mathcal{L}_{qD}^j)\right)^G \qquad \text{and} \qquad \operatorname{Proj} R_D \cong \operatorname{Proj} R_{qD} \quad .$$

Let $x_1^{a_1}\dots x_n^{a_n}$ be a monomial in R_{qD}. There is a positive integer m such that

$$a_1C_1+\cdots+a_nC_n=mqD \qquad .$$

In particular, $a_1a_{11}=q\alpha$ for some non-zero integer α, so q divides a_1. Therefore x_1 appears in R_{qD} only to the q^{th} power as x_1^q. Hence

$$R_{qD}\cong\bigoplus_{j=0}^{\infty}H^0(\operatorname{Spec}(\mathbb{C}[x_1^q,x_2,\dots,x_n]),\mathcal{L}_D^j)^G \qquad ,$$

but this is the coordinate ring of $\mathfrak{X}(I,B)$. □

Definition 2.10. The quotient space, or the toric stack, defined by $\mathfrak{X}(I,A)$ is called *well formed* if its defining matrix A is well formed.

Corollary 2.11. *For any quotient $\mathfrak{X}(I,A)$ there exists a well formed model $\mathfrak{X}(I,B)$, which is a toric variety. Moreover, the quotient spaces $\mathfrak{X}(I,A)\cong\mathfrak{X}(I,B)$, as schemes, and $\mathfrak{X}(I,B)$ is the canonical model of the stack $\mathfrak{X}(I,A)$.*

Remark 3. Note that $\mathfrak{X}(I,B)$ is the coarse moduli space for a toric Deligne-Mumford stack $\mathfrak{X}(I,A)$, where B is the standard matrix of A.

Example 2.12. Consider the following matrix:

$$A=\begin{pmatrix}1&1&1&0&-2\\1&1&1&2&0\end{pmatrix}$$

This matrix is not standard as $d_A=2$. The standard (well formed) model can be obtained by multiplying the following 2×2 matrix

$$\begin{pmatrix}1&0\\-1&1\end{pmatrix}\in\mathrm{SL}^*(2,\mathbb{Z})$$

and then removing the factor of 2 from the second row, which results in

$$\begin{pmatrix}1&1&1&0&-2\\0&0&0&1&1\end{pmatrix}$$

The geometry of this example: Consider the weighted projective space $\mathbb{P}=\mathbb{P}(1,1,1,2)$. This is an orbifold with a terminal cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$. Consider eigencoordinates x,y,z,t on $\mathbb{P}$. The projective space $\mathbb{P}$ is covered by 4 open affine patches, three of them are $U_x\cong U_y\cong U_z\cong\mathbb{C}^3$, where U_x, for example, is the Zariski open subset $x\neq 0$, and the fourth patch is $U_t=C^3/\!/\mathbb{Z}_2$. The action of Z_2 on $\mathbb{C}^3$ is given by $(\bar{x},\bar{y},\bar{z})\mapsto(\epsilon\bar{x},\epsilon\bar{y},\epsilon\bar{z})$, for ϵ a second root of unity, and is traditionally denoted by $\frac{1}{2}(1,1,1)$.

Let us explain the toric structure of $\mathbb{P}$. The fan consists of 4 rays:

$$r_1=(1,0,0),\quad r_2=(0,1,0)$$

$$r_3=(0,0,1),\quad r_4=(-1,-1,-2)$$

and they generate 4 maximal cones:

$$C_1=\langle r_1,r_2,r_3\rangle\,,\;C_2=\langle r_1,r_3,r_4\rangle\,,\;C_3=\langle r_1,r_2,r_4\rangle\ \text{and}\ C_4=\langle r_1,r_2,r_3\rangle$$

one can associate y,z,t,x with r_1,r_2,r_3 and r_4, in that order, and check the structure of the cones, in particular the fact that C_1,C_2 and C_4 are smooth and C_3 is the singular cone. In order to resolve this singularity it is enough to introduce a new ray $r_5=(0,0,-1)$ and do the corresponding subdivision. One can compute the Cox ring of this new toric variety (using techniques in [11] and [12]) to see that this variety is the quotient of $\mathbb{C}^5-Z(I)$ by $(\mathbb{C}^*)^2$, where

I, the irrelevant ideal, is $I = (x_1, x_2, x_3) \cap (x_4, x_5)$ when $x_1, \dots, x_5$ are the eigencoordinates on $\mathbb{C}^5$, and the action is induced by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In other words, this rank 2 toric variety is the weighted blow up of $\mathbb{P}(1,1,1,2)$ at the singular point. For a reader interested in minimal model theory of toric varieties: if one requires to carry out the 2-ray game (as in the Sarkisov program [7]), starting from this blow up, will see (using techniques in [4]) that the other end results in a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

3. Blow ups of low rank Cox rings and Sarkisov links

We begin this section by considering a special class of rank 2 toric varieties. The goal is to understand their singularities and to construct explicit tools to write down the Cox rings of their (toric, weighted) blow ups. In [4] it was shown how the Cox data of a rank 2 Cox ring changes as one runs the 2-ray game to obtain type III or IV Sarkisov links. Another goal in this section is to understand what happens, in terms of Cox data, when one runs the 2-ray game by starting a blow up on the rank 2 Cox ring, which, in good situations, leads to type I or II Sarkisov links.

Definition 3.1. A *weighted bundle over* $\mathbb{P}^n$ is a rank 2 toric variety $\mathcal{F} = \mathfrak{X}(A, I)$ defined by

(i) $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$.
(ii) The irrelevant ideal of $\mathcal{F}$ is $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.
(iii) and the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{n+m+2}$ is given by

$$A = \begin{pmatrix} 1 & \dots & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad ,$$

where ω_i are non-negative integers and $\mathbb{P}(1, a_1, \dots, a_m)$ is a well formed weighted projective space.

We denote this quotient by $\mathcal{F}_{\mathbb{P}}(\omega_0, \dots, \omega_m)$ or sometimes simply by $\mathcal{F}$, when there is no ambiguity.

Note that the weighted bundle $\mathcal{F}$ defined in Definition 3.1 is well formed because the weighted projective space $\mathbb{P}(1, a_1, \dots, a_m)$ is well formed.

The following lemma constructs the fan associated to the weighted bundle in Definition 3.1.

Theorem 3.2. *Let $\beta_1, \dots, \beta_m, \alpha_1, \dots, \alpha_n$ be the standard basis of $\mathbb{Z}^{n+m}$. Suppose α_0 and β_0 are the following vectors in $\mathbb{Z}^{n+m}$.*

$$\beta_0 = -\sum_{i=1}^{m} a_i \beta_i \quad , \quad \alpha_0 = -\sum_{j=1}^{n} \alpha_j + \sum_{i=0}^{m} \omega_i \beta_i \quad ,$$

where ω_i are non-negative integers. Let $\sigma_{rs} = \left\langle \beta_0, \dots, \hat{\beta}_r, \dots, \beta_m, \alpha_0, \dots, \hat{\alpha_s}, \dots, \alpha_n \right\rangle$ be the cone in $\mathbb{Z}^{n+m}$ generated by $\beta_0, \dots, \hat{\beta}_r, \dots, \beta_m$ and $\alpha_0, \dots, \hat{\alpha_s}, \dots, \alpha_n$, where α_s and β_r are omitted. If we denote Σ for the fan in $\mathbb{Z}^{n+m}$ generated by maximal cones σ_{rs} for all $0 \le r \le n$ and $0 \le s \le m$, then $\mathcal{F} \cong T(\Sigma)$.

Proof. We compute the GIT construction of this fan following the recipe of Cox given in [12] §10. By assumption, rays $\alpha_0, \dots, \alpha_n, \beta_0, \dots, \beta_m$ in $N = \mathbb{Z}^{m+n}$ form $\Delta(1)$, the set of 1-dimensional cones in Σ. Let us associate the variables $x_0, \dots, x_n, y_0, \dots, y_m$ to these rays. For a given maximal cone σ, define x^σ to be the product of all variables not coming from

rays of σ. But maximal cones in Σ are exactly σ_{rs}, which immediately implies $x^{\sigma_{rs}} = x_s y_r$. The irrelevant ideal is given by

$$I = (x^\sigma \mid \sigma \in \Sigma \text{ is a maximal cone}) = (x_s y_r \mid 0 \le s \le n \text{ and } 0 \le r \le r) \quad .$$

so that $I = (x_0, \dots, x_n) \cap (y_0, \dots, y_m)$.

In order to describe the GIT construction of $T(\Sigma)$ we must find the group G such that

$$T(\Sigma) \cong (\operatorname{Spec}[x_0, \dots, x_n, y_0, \dots, y_m] - V(I))/G \quad ,$$

where $G \subset (\mathbb{C}^*)^{m+n+2}$ is defined by

$$G = \{(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in (\mathbb{C}^*)^{m+n+2} \mid \prod_{i=0}^{n} \mu_i^{\langle e_k, \alpha_i \rangle} . \prod_{j=0}^{m} \lambda_j^{\langle e_k, \beta_j \rangle} = 1, \text{ for all } k\} \quad ,$$

and $e_1, \dots, e_{m+n}$ form the standard basis of $\mathbb{Z}^{m+n}$. But the standard basis of $\mathbb{Z}^{m+n}$, by assumption, is $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m\}$.

Computing this set implies that $(\mu_0, \dots, \mu_n, \lambda_0, \dots, \lambda_m) \in G$ if and only if

$$\mu_i . \mu_0^{\langle \alpha_0, \alpha_i \rangle} . \lambda_0^{\langle \beta_0, \alpha_i \rangle} = 1 \quad \text{and} \quad \lambda_j . \mu_0^{\langle \alpha_0, \beta_j \rangle} . \lambda_0^{\langle \beta_0, \beta_j \rangle} = 1 \quad \forall i, j \quad .$$

In other words, λ_0 and μ_0 determine all other λ_j and μ_i. Therefore the group G is isomorphic to $(\mathbb{C}^*)^2$ and the action on coordinate variables is defined by

$$((\mu, \lambda).x_0) = \mu x_0 \qquad ((\mu, \lambda).x_i) = \mu^{-\langle \alpha_0, \alpha_i \rangle} \lambda^{-\langle \beta_0, \alpha_i \rangle} x_i \quad ,$$
$$((\mu, \lambda).y_0) = \lambda y_0 \qquad ((\mu, \lambda).y_j) = \mu^{-\langle \alpha_0, \beta_j \rangle} \lambda^{-\langle \beta_0, \beta_j \rangle} y_j \quad .$$

In other words, $(\mathbb{C}^*)^2$ acts on $\mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m]$ by the matrix

$$B = \begin{pmatrix} 1 & \dots & 1 & 0 & \omega_0 a_1 - \omega_1 & \dots & \omega_0 a_m - \omega_m \\ 0 & \dots & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad .$$

We have shown so far that $T(\Sigma) \cong \mathfrak{X}(B, I)$. Multiplying B on the left by the matrix

$$\begin{pmatrix} 1 & -\omega_0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}^*(2, \mathbb{Z}) \quad ,$$

together with Lemma 2.3 proves that $\mathcal{F} \cong T(\Sigma)$. □

Remark 4. In [21] Chapter 2, Reid gives a detailed analysis of rational scrolls, which, in our setting, are the weighted bundles over $\mathbb{P}^1$, with weights 1 only. In fact these are the smooth weighted bundles.

Proposition 3.3. *A well formed weighted bundle $\mathcal{F}$, defined in Definition 3.1, is covered by $(n+1)(m+1)$ patches, each of them isomorphic to a quotient of $\mathbb{C}^{n+m}$ by a cyclic group $\mathbb{Z}_k$, for some positive integer k.*

Proof. We construct the patches $\mathcal{U}_{ij}$ for $0 \le i \le n$ and $0 \le j \le m$. Note that at the toric level, $\mathcal{U}_{ij} = (x_i y_j \neq 0)$ corresponds to the maximal cone σ_{ij} as in Proposition 3.2.

$$\mathcal{U}_{ij} = \operatorname{Spec} \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_m, x_i^{-1}, y_j^{-1}]^{\mathbb{C}^* \times \mathbb{C}^*}$$

Computing the invariants gives

$$\mathcal{U}_{ij} = \operatorname{Spec} \mathbb{C}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0^{a_j}}{y_j} . x_i^{\omega_0 a_j - \omega_j}, \dots] \quad .$$

Again powers of x_i appear to make each term invariant under the action of the first coordinate of $(\mathbb{C}^*)^2$ and each y_k comes with a power that is the first number which is 0 modulo a_j. In other words, these invariants are exactly the same as those of $\frac{1}{a_j}(0, \dots, 0, 1, a_1, \dots, a_n)$. □

3.1. **Blow ups of weighted projective space.** Example 2.12 already explained the blow up of the weighted projective space $\mathbb{P}(1,1,2)$ at its singular point. In general the rank two toric variety T (or stack, if not well-formed) defined by the homogeneous coordinate ring $C[y, x_0, \dots, x_n]$ and the irrelevant ideal $I = (y, x_0, \dots, x_k) \cap (x_{k+1}, \dots, x_n)$ and the weight system (indicating the action of $(\mathbb{C}^*)^2$)

$$\begin{pmatrix} \alpha & 0 & \dots & 0 & -b_{k+1} & \dots & -b_n \\ 0 & a_0 & \dots & a_k & a_{k+1} & \dots & a_n \end{pmatrix} \quad ,$$

for $1 \leq k \leq n-2$ is the (weighted) blow up of the centre $X : (x_{k+1} = \dots = x_n = 0) \cong \mathbb{P}(a_0, \dots, a_k) \subset \mathbb{P}(a_0, \dots, a_n)$. Details of this constructions are left to the reader to check. A more complicated situation is explained in the next part and the idea and techniques of the proofs there can be applied to this case.

The 2-ray game of this space follows the variation of GIT as explained in [4]. As an illustration, the following example shows how this can be used to study birational geometry of hypersurfaces of weighted projective spaces. Consider a Fano threefold X that is a general hypersurface of degree 24 in $\mathbb{P} = \mathbb{P}(1,1,6,8,9)$. In a suitable coordinate system, X is defined by $\{f = x_5^2 x_3 + x_4^3 + x_3^4 + \dots + x_1^{24}\}$ in $\mathbb{P}$. Suppose $p_5 = (0:0:0:0:1)$. The germ $p_5 \in \mathbb{P}$ is of type $1/9(1,1,6,8)$ and the germ $p_5 \in X$ is a terminal quotient singularity of type $1/9(1,1,8)$. Consider T the blow up of this point in the above description. T is given by a rank two variety with weight system

$$\begin{pmatrix} 3 & 0 & -2 & -6 & -1 & -1 \\ 0 & 9 & 8 & 6 & 1 & 1 \end{pmatrix}$$

with the coordinate system $\mathbb{C}[u, x_5, x_4, x_3, x_2, x_1]$ and the irrelevant ideal $I = (u, x_5) \cap (x_4, x_3, x_2, x_1)$. In other words the blow up is given by $(x_5, x_4 u^{\frac{2}{3}}, x_3 u^2, x_2 u^{\frac{1}{3}}, x_1 u^{\frac{1}{3}})$, in coordinates. The restriction of this toric construction to X is the blow up $\hat{X} \to X$ with weights $1/3(1,1,2)$ blow up of $p_5 \in X$. One can check that T is not well-formed and its well-formed model has the weight system

$$\begin{pmatrix} 1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 9 & 8 & 6 & 1 & 1 \end{pmatrix}$$

with respect to which, the equation of $\hat{X}$ has bi-degree $(6,24)$, with equation $x_3 x_5^2 + x_4^3 + x_3^4 u^6 + \dots + u^6 x_1^3$. Following the 2-ray game of the ambient space (using techniques of [4]), it is easy to verify that $\hat{X}$ forms a fibration over $\mathbb{P}(1,1,6)$ with elliptic fibres $E_6 \subset \mathbb{P}(1,2,3)$. In [6], Cheltsov and Park study elliptic and K3 fibrations of Fano threefolds by considering a projection and looking at the local equation of the fibres. As shown in the example above their results can be recovered by our methods by global calculations.

3.2. **Blow ups of rank 2 toric varieties.** Now we construct Cox rings of rank 3, obtained by blowing up some centres in a rank 2 toric variety. Then we try to understand the nature of the maps from these varieties to the rank 2 ones. We do this on weighted bundles over $\mathbb{P}^1$, i.e. when the coordinate ring is $\mathbb{C}[x_0, x_1, y_0, \dots, y_n]$ with irrelevant ideal $I = (x_0, x_1) \cap (y_0, \dots, y_n)$ and weight system

$$\begin{pmatrix} 1 & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_m \\ 0 & 0 & 1 & a_1 & \dots & a_m \end{pmatrix} \quad ,$$

for positive integers $a_1, \dots, a_n$ and non-negative integers $\omega_0, \dots, \omega_n$.

It was shown in Proposition 3.3 that each germ

$$p_{rs} = \{x_i = y_j = 0; \forall\, (i.j) \neq (r,s)\} \in \mathcal{U}_{rs} = \{x_r x_s \neq 0\}$$

has a cyclic quotient singularity of type $\frac{1}{a_j}(0,1,\dots,a_m)$. Of course this singularity is not isolated. However, instead of blowing up the orbifold locus, we blow up a closed point. The

reason for doing this is that we often want to consider the blow up of some subvarieties of $\mathcal{F}$ only at a particular point, see next section for an illustration. We do this by considering the blow up of the ambient space at this point and restrict our attention to the subvariety under this blow up.

Fix $k \in \{0, \dots, m\}$ and let T be a rank 3 toric variety defined by

(i) $\mathrm{Cox}(T) = \mathbb{C}[X_0, X_1, Y_0, \dots, Y_m, \xi]$,

(ii) the irrelevant ideal

$$J = (X_0, X_1) \cap (Y_0, \dots, Y_m) \cap (\xi, X_1) \cap (\xi, Y_k) \cap (X_0, Y_0, \dots, \hat{Y_k}, \dots, Y_m) \quad \text{and}$$

(iii) the action of $(\mathbb{C}^*)^3$ given by the matrix

$$\begin{pmatrix} 1 & 1 & -\omega_0 & -\omega_1 & \dots & -\omega_{k-1} & -\omega_k & -\omega_{k+1} & \dots & -\omega_m & 0 \\ 0 & 0 & 1 & a_1 & \dots & a_{k-1} & a_k & a_{k+1} & \dots & a_m & 0 \\ b_k & 0 & b_0 & b_1 & \dots & b_{k-1} & 0 & b_{k+1} & \dots & b_m & -a_k \end{pmatrix} ,$$

where $b_0, \dots, b_m$ are strictly positive integers such that

$$b_i \equiv a_i \mod a_k \text{ for } i \neq k \qquad \text{and} \qquad b_k = r a_k \text{ for some positive integer } r \quad .$$

Proposition 3.4. *The rank 3 toric stack T constructed above is the blow up of the weighted bundle $\mathcal{F}$ over $\mathbb{P}^1$ in Definition 3.1 at the point* $(0 : 1; 0 : \dots : 0 : 1 : 0 \dots : 0)$.

Proof. By Proposition 3.2, the fan associated to $\mathcal{F}$ consists of 1-dimensional cones β_0, β_1 and $\alpha_0 \dots, \alpha_m$ in $N = \mathbb{Z}^{m+1}$ with $2m + 2$ maximal cones

$$\sigma_{0i} = \langle \beta_1, \alpha_0 \dots, \hat{\alpha_i}, \dots, \alpha_m \rangle \text{ and } \sigma_{1j} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha_j}, \dots, \alpha_m \rangle \text{ for } 0 \leq i, j \leq m \quad .$$

The last row of the defining matrix of T is clearly adding a new ray in the cone σ_{0k}. The fact that the generator of this ray is an integral vector in N is guaranteed by the conditions imposed on b_i. This implies that T is the blow up of $\mathcal{F}$ at a point if it has the correct irrelevant ideal. We complete the proof by showing the irrelevant ideal of the Cox ring of this toric blow up is precisely the ideal of T. This is done by taking the subdivision of σ_{0k} and computing the irrelevant ideal of the new fan using the method of [12], as in the proof of Proposition 3.2. The fan of this blow up of Σ consists of rays $\beta_0, \beta_1, \alpha_0, \dots, \alpha_m, \gamma$ and maximal cones

$$\sigma'_{ki} = \langle \beta_0, \alpha_0 \dots, \hat{\alpha_i}, \dots, \hat{\alpha_k}, \dots, \alpha_m, \gamma \rangle \text{ for } i \neq k \text{ and } \sigma'_k = \langle \alpha_0 \dots, \hat{\alpha_k}, \dots, \alpha_m, \gamma \rangle$$

coming from the subdivision of σ_{0k} together with the remaining cones σ_{ij}. If we associate the new variable ξ to the ray γ and X_0, X_1 to β_0, β_1 and Y_i to α_i, then the irrelevant ideal of this toric variety is the ideal generated by

$$X_1 \cdot Y_i \cdot Y_k \text{ for all } i \neq k, \quad X_0 \cdot X_1 \cdot Y_k, \quad X_1 \cdot Y_i \cdot \xi \text{ for all } i \neq k, \quad X_0 \cdot Y_i \cdot \xi \text{ for all } i \quad .$$

The primary decomposition of this ideal is the irrelevant ideal of T. □

4. Birational models of the del Pezzo fibration

Our initial model X is defined as a complete intersection of two hypersurfaces in $\mathcal{F}$, where $\mathcal{F}$ is a $\mathbb{P}^4$-bundle over $\mathbb{P}^1$. If we denote by $y_0, y_1, x_0, \cdots, x_4$, the global coordinates on $\mathcal{F}$, then the Cox ring of $\mathcal{F}$ is $\mathrm{Cox}(\mathcal{F}) = \mathbb{C}[y_0, y_1, x_0, \cdots, x_4]$ with weights

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -3 & -3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} .$$

and irrelevant ideal $I = (y_0, y_1) \cap (x_0, \cdots, x_4)$.

4.1. **Cox ring and description of the initial model.** Let $Q_1 = \{f = 0\}$ and $Q_2 = \{g = 0\}$, where $f \in |\mathcal{O}(-3,2)|$, i.e. f has bi-degree $(-3,2)$, and $Q_2 \in |\mathcal{O}(-2,2)|$. Assume furthermore that f has no monomial term x_0x_4 and similarly g has no $y_0x_0x_3$ or $y_1x_0x_3$, and otherwise f and g are general.

Lemma 4.1. *$X = Q_1 \cap Q_2 \subset \mathcal{F}$ is smooth.*

Proof. A simple calculation on the Jacobian matrix of X shows that $\mathrm{Sing}(X) \subset \mathbb{P}^1_0$, where $\mathbb{P}^1_0 = (x_1 = x_2 = x_3 = x_4 = 0)$. Having monomials of type $x_0 \times l$, where l is a linear term, in the equation of X imply smoothness along this line. □

Proposition 4.2. $\mathrm{Cox}(X) = \dfrac{\mathrm{Cox}(\mathcal{F})}{(f,g)}$.

Proof. It is easy to check, using methods in [1] §4.3 that $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathcal{F})$. Factoriality of $\mathcal{F}$ together with Lemma 4.1, imply that $X \subset \mathcal{F}$ is a neat embedding, see [17] Definition 2.5. The result follows from [17] Corollary 2.7. □

4.2. **Different Mori structures.** Now we show how the other Mori fibre space models, birational to X, are obtained.

By rules of Sarkisov program, see [8] §2.2, there are two possibilities for the start of the program: having a Mori fibre space $X \to S$, either run one step of the MMP on S, obtain $S \dashrightarrow T$, and consider $\mathrm{Pic}(X/T)$ or do a blow up on X, obtain $Z \to X$, and consider $\mathrm{Pic}(Z/S)$. In both cases the relative Picard group has rank 2. One generator of this group corresponds to $Z \to X$ or $S \dashrightarrow T$ and the other generator indicates the beginning of the so-called 2-ray game, and in correct setting the Sarkisov link. See [8] for details. Running the first type is most commonly used for varieties constructed in similar ways to our objects, see for example [1,3]. The general setting for this kind of links is described in [4], using Variation of Geometric Invariant Theory (vGIT). Following these techniques one can see that X is birational to Y, a del Pezzo fibration of degree 2 over $\mathbb{P}^1$. The link between X and Y consists of an anti-flip, of local type $(1,1,-1,-3)$ followed by an Atiyah flop.

Proposition 4.3. *X is birational to a del Pezzo fibration of degree 2 over $\mathbb{P}^1$.*

The Cox ring of Y is $\mathrm{Cox}(Y) = \mathbb{C}[u,v,x,y,z,t,s]/(f,g)$, with weights

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{pmatrix},$$

and irrelevant ideal $I_Y = (u,v) \cap (x,y,z,t,s)$.

Note that, for simplicity, we have renamed the variables, i.e., the variables u,x,y,z,t,s are exactly $x_4,x_3,x_2,x_1,x_0,y_0,y_1$, in that order. The weight matrix is that of $\mathcal{F}$ in opposite order, in rays and columns, multiplied by a matrix

$$\begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix} \in \mathrm{SL}^*(2,\mathbb{Z}).$$

The equations of f and g must be easily read after the substitution, in particular, $f = 0.vz + uz + x^2t + \cdots$ and $g = 0.vzt + 0.vzs + y^2 + xz + \cdots$, with bi-degrees $(-1,3)$ and $(-2,4)$, with respect to the new weights. By Proposition 3.3, $\mathcal{F}_1$, the ambient toric variety in which Y is embedded, has two lines of singularities of types $\mathbb{A}^1 \times 1/2(1,1,1,1)$ and $\mathbb{A}^1 \times 1/3(1,1,1,2)$. It is easy to check that Y does not meet the first line and it intersects the second line at the point $p_{vz} = (u = x = y = t = s = 0)$. In particular, p_{vz} is a terminal singularity of type $1/3(1,1,2)$, as one can eliminate the variables u and x in an analytical neighbourhood of this point using f and g.

We want to show that a blow up of the point p_{vz} is the start of a Sarkisov link on Y. By [19] this blow up is the unique weighted blow up of type $(1,1,2)$ with discrepancy $\frac{1}{3}$. In other words, if we denote the blow up of Y by $\mathcal{Y}$, and the exceptional divisor of the blow up by E, then

$$K_{\mathcal{Y}} = K_Y + \frac{1}{3}E$$

Define a rank 3 toric Cox ring by $\mathcal{R} = \mathbb{C}[u,v,x,y,z,t,s,w]$, the irrelevant ideal $\mathcal{I} = (u,v) \cap (x,y,z,t,s) \cap (u,x,y,t,s) \cap (w,v) \cap (w,z)$ and weights given by the matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 0 & a & 2 & 0 & 1 & 1 & -3 \end{pmatrix}.$$

By Proposition 3.4, $\mathcal{T} = T(\mathcal{I}, \mathcal{A})$ is a blow up of $\mathcal{F}_1$, for a positive integer $a = 3k+1$. The aim is to show that for a particular k, $\mathcal{Y}$ is neatly embedded in $\mathcal{T}$.

Note that the blow up map is given by

$$\varphi((u,v,x,y,z,t,s,w)) \mapsto (uw, v, w^{\frac{a}{3}}x, w^{\frac{2}{3}}y, z, w^{\frac{1}{3}}t, w^{\frac{1}{3}}s)$$

The well formed model of $\mathcal{A}$ is

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & k & 0 & -1 & 0 & 0 & -1 \end{pmatrix}.$$

Lemma 4.4. *For $a = 4$, $\mathcal{Y} \longrightarrow Y$ is the Kawamata blow up of the point $p_{vz} \in Y$, where $\mathcal{Y} \subset \mathcal{T}$ is the birational transform of Y by φ.*

Proof. Since Y is a complete intersection in T, we have that $-K_Y = (-K_T - Y)|_Y$, and similar formula for $\mathcal{Y} \subset \mathcal{T}$. If we denote by $\tilde{f}$ and $\tilde{g}$ the defining equations of $\mathcal{Y} \in \mathcal{T}$, then $\deg \tilde{f} = (-1,3,0)$ and $\deg \tilde{g} = (-2,4,0)$. In particular,

$$-K_Y \sim \mathcal{O}_Y((0,1)) \quad \text{and} \quad -K_{\mathcal{Y}} \sim \mathcal{O}_{\mathcal{Y}}((0,1,k-1))$$

In other words, $K_{\mathcal{Y}} \sim -D_x - E$, where $D_x = (x=0)$ is a principal divisor on $\mathcal{Y}$ and $E = (w=0)$ is the exceptional divisor. On the other hand, $\varphi^*(K_Y) \sim -D_x - \frac{a}{3}E$. Using Kawamata's condition that the discrepancy is equal to $\frac{1}{3}$, we conclude that $a = 4$. $\square$

Lemma 4.5. *Y is square birational to a degree 2 del Pezzo model neatly embedded in a toric variety as a hypersurface.*

Proof. Using toric MMP we can see that the 2-ray game on $\mathcal{T}$, associate to the rank 2 relative Picard group of $\mathcal{T}/\mathbb{P}^1_{u:v}$, consists of a flop to $\mathcal{T}'$ followed by a divisorial contraction to a rank 2 toric variety $\mathcal{F}_2$. This is done by fixing the action of the first component of the $(\mathbb{C}^*)^3$, by the choice of a nonzero value for v (as we work over the field of rational functions of $\mathbb{P}^1$), and then reducing the problem to a rank 2 toric 2-ray game, over this field. The flop is the contraction of the lines (fibres) in a $\mathbb{P}^1$-bundle of $\mathbb{P}(1,1,2)$ on one side and extracting another $\mathbb{P}^1$-bundle on the other side of the flop. In terms of Cox rings, this is the replacement of $\mathcal{I}$ by another ideal $\mathcal{I}' = (u,v) \cap (w,v) \cap (u,x) \cap (w,y,z,t,s) \cap (x,y,z,t,s)$. If we rewrite $\mathcal{A}$, after a $\mathrm{SL}^*(3,\mathbb{Z})$ modification, as

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 & -1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 0 & 0 & -2 & -4 & -1 & -1 & -1 \end{pmatrix}.$$

By Proposition 3.4, $\mathcal{T}' \longrightarrow \mathcal{F}_2$ is a divisorial contraction with exceptional divisor $E' = (u = 0)$ to a point, where $\mathrm{Cox}(\mathcal{F}') = \mathbb{C}[w, v, z, y, x, t, s]$, with irrelevant ideal $I_2 = (w, v) \cap (z, y, x, t, s)$ and the weights

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 3 & 2 & 1 & 1 & 1 \end{pmatrix}.$$

We can write down the equation of this contraction by

$$\psi((u, v, x, y, z, t, s, w)) \mapsto (uw, v, u^4 z, u^2 y, x, ut, us),$$

in particular, the image of the contraction is the point $p_{vx} \in \mathcal{F}_2$.

Restricting all these to $\mathcal{Y}$, we see that the flop of the ambient space restricts to a flop of 6 lines, associate to the intersection of a quartic, coming from the birational transform of g, and a cubic, coming from f, in $\mathbb{P}(1, 1, 2)$, that is 6 points (see [18], Lemma 9.5). The exceptional divisor is a cubic in $\mathbb{P}(1, 1, 2)$. The birational transforms of f and g in $\mathcal{F}_2$ are, respectively $\hat{f} = z + wxy + vyt + \cdots$ and $\hat{g} = y^2 + xz + \cdots$. In particular, we can eliminate z globally, and consider Z, the birational transform of Y, embedded as a hypersurface in a rank 2 toric variety with Cox ring equal to that of $\mathcal{F}_2$, with z eliminated. Z is the vanishing of the polynomial $y^2 = vyxt + w^2 t^4 + \cdots$. It is easy to check that Z is a fibration of degree 2 del Pezzo surfaces over $\mathbb{P}^1_{v:w}$, and is square birational to $Y/\mathbb{P}^1$. □

Lemma 4.6. *Z is birational to a conic bundle.*

Proof. This appears as Family 6 in Theorem 3.3 of [1]. □

The following diagram shows the geometry of X and its birational models.

$$\begin{tikzcd}
 & & & \mathcal{Y} \arrow[r, dashed, "6\times(1,1,-1,-1)"] \arrow[dl, "\text{blow up}"'] & \mathcal{Y}' \arrow[dr, "\text{blow up}"] & & \\
X \arrow[r, dashed, "(1,1,-1,-3)"] \arrow[d, "dP_4"'] & X_1 \arrow[r, dashed, "(1,1,-1,-1)"] & Y \arrow[dr, "dP_2"] & & & Z \arrow[r, "\simeq"] \arrow[dl, "dP_2"'] & Z \arrow[d, "\text{conic}"', "\text{bundle}"] \\
\mathbb{P}^1 & & & \mathbb{P}^1 \arrow[r, dashed, "\simeq"] & \mathbb{P}^1 & & \mathbb{P}^2
\end{tikzcd}$$

In order to complete the proof of Theorem 1.3 we need to show that X is not rational. We use the result of Alexeev [2] to obtain this.

Theorem 4.7 ([2], Theorem 2)**.** *If the Euler characteristic of a (standard) del Pezzo fibration of degree 4 over $\mathbb{P}^1$ does not belong to $\{0, -4, -8\}$, then it is not rational.*

Lemma 4.8. *X is not rational.*

Proof. By the same calculation as [15] Example 3.2.11 we can compute $\chi(X) = -28$, the Euler characteristic of X. The result follows from Theorem 4.7. □

Some experience in working with del Pezzo fibrations, and the observations above, lead to the following natural conjecture.

Conjecture 4.9. Let X be a smooth threefold Mori fibre space over $\mathbb{P}^1$, with fibres being smooth del Pezzo surfaces of degree 4. Then, except the birational maps obtained by blowing up a section, there is no Sarkisov link starting from X to another Mori fibre space if and only if $-K_X \notin \mathrm{Int}(\overline{\mathrm{Mob}(X)})$.

References

1. Hamid Ahmadinezhad, *On del Pezzo fibrations that are not birationally rigid*, J. Lond. Math. Soc. **86** (2012), no. 1, 36–62.
2. V. A. Alekseev, *On conditions for the rationality of three-folds with a pencil of del Pezzo surfaces of degree* 4, Mat. Zametki **41** (1987), no. 5, 724–730, 766. MR 898133 (88j:14048)
3. Gavin Brown, Alessio Corti, and Francesco Zucconi, *Birational geometry of 3-fold Mori fibre spaces*, The Fano Conference, Univ. Torino, Turin, 2004, pp. 235–275. MR 2112578 (2005k:14031)
4. Gavin Brown and Francesco Zucconi, *Graded rings of rank 2 Sarkisov links*, Nagoya Math. J. **197** (2010), 1–44. MR 2649280 (2011f:14020)
5. Ivan Cheltsov, *Birationally rigid del Pezzo fibrations*, Manuscripta Math. **116** (2005), no. 4, 385–396. MR 2140210 (2006b:14067)
6. Ivan Cheltsov and Jihun Park, *Weighted Fano threefold hypersurfaces*, J. Reine Angew. Math. **600** (2006), 81–116. MR 2283799 (2007k:14019)
7. Alessio Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), no. 2, 223–254. MR 1311348 (96c:14013)
8. ———, *Singularities of linear systems and* 3*-fold birational geometry*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 259–312. MR 1798984 (2001k:14041)
9. Alessio Corti and Massimiliano Mella, *Birational geometry of terminal quartic 3-folds. I*, Amer. J. Math. **126** (2004), no. 4, 739–761. MR 2075480 (2005d:14019)
10. Alessio Corti, Aleksandr Pukhlikov, and Miles Reid, *Fano* 3*-fold hypersurfaces*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 175–258. MR 1798983 (2001k:14034)
11. David A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR 1299003 (95i:14046)
12. ———, *What is a toric variety?*, Topics in algebraic geometry and geometric modeling, Contemp. Math., vol. 334, Amer. Math. Soc., Providence, RI, 2003, pp. 203–223. MR 2039974 (2005b:14089)
13. Igor Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71. MR 704986 (85g:14060)
14. Barbara Fantechi, Etienne Mann, and Fabio Nironi, *Smooth toric Deligne-Mumford stacks*, J. Reine Angew. Math. **648** (2010), 201–244. MR 2774310 (2012b:14097)
15. William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037 (94g:14028)
16. M. M. Grinenko, *Fibrations into del Pezzo surfaces*, Uspekhi Mat. Nauk **61** (2006), no. 2(368), 67–112. MR 2261543 (2007g:14009)
17. Jürgen Hausen, *Cox rings and combinatorics. II*, Mosc. Math. J. **8** (2008), no. 4, 711–757, 847. MR 2499353 (2010b:14011)
18. A. R. Iano-Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds, London Math. Soc. Lecture Note Ser., vol. 281, Cambridge Univ. Press, Cambridge, 2000, pp. 101–173. MR 1798982 (2001k:14089)
19. Yujiro Kawamata, *Divisorial contractions to* 3*-dimensional terminal quotient singularities*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 241–246. MR 1463182 (98g:14005)
20. A. V. Pukhlikov, *Birational automorphisms of three-dimensional algebraic varieties with a pencil of del Pezzo surfaces*, Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), no. 1, 123–164. MR 1622258 (99f:14016)
21. Miles Reid, *Chapters on algebraic surfaces*, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159. MR 1442522 (98d:14049)
22. K. A. Shramov, *On the rationality of nonsingular threefolds with a pencil of del Pezzo surfaces of degree 4*, Mat. Sb. **197** (2006), no. 1, 133–144. MR 2230135 (2007h:14015)

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